SAINSTEK: JURNAL SAINS DAN TEKNOLOGI
Publisher: AMSET IAIN Batusangkar and IAIN Batusangkar Press
Website: http://ojs.iainbatusangkar.ac.id/ojs/index.php/sainstek
E-mail: sainstek@iainbatusangkar.ac.id

December 2023
Vol 15 No 2
ISSN: 2085-8019 (p)
ISSN: 2580-278X (e)
pp : 98-109

Stability Analysis of Fixed Points in Forest Biomass Depletion Model

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Article History
Received: 10 May 2023
Reviewed: 2 August 2023
Accepted: 14 August 2023
Published: 31 December 2023

Key Words
Fixed points; Forest biomass; Population; Stability.

Abstract
The main focus of this article is to present a model that explains the depletion in forest biomass (B) due to population size (N), population pressure (P), and industrialization (I). This research explores the complex relationship between forest biomass, population growth, and industrialization. The model generates four fixed points that are all non-negative, and they are known as $E_1, E_2, E_3$and E^*. The article proceeds to analyze the stability of these fixed points in the context of forest biomass depletion. It is discovered that the fixed points E_1, E_2, and $E_3$are saddle points and not stable, while the fixed point E^* is stable if it meets certain conditions. The article concludes by carrying out numerical simulations to determine the equilibrium point of the model, which shows that forest resource biomass stability declines as population size, population pressure, and industrialization increase. The simulations reveal that population growth results in a depletion in forest resource biomass, while the opposite is true for the positive impact that forest resource biomass on population levels. Consequently, it is essential to regulate population density and industrialization to manage population growth and protect forest resources.

INTRODUCTION

Forests have an important role in preserving the ecological balance of the planet. They play a crucial role in regulating the exchange of oxygen and carbon dioxide. Trees in forests emit oxygen, which helps in purifying and refreshing the air, and they also take in carbon dioxide, storing it as biomass. Furthermore, forests act as reservoirs of water, provide raw materials for industries, fuel sources, and are also ideal for recreational activities.

The numerous benefits of forests for supporting human life have led to forest exploration. However, it is necessary for humans to be aware of the importance of protecting the environment and forests. Uncontrolled exploration leads to a depletion in forest area, as well as forest fires caused by conversion to agricultural land and settlements, accelerating the decrease in forest area. This occurs in developing countries where the depletion in forest area is caused by the excessive use of forests by the population for infrastructure and industrial development (Dubey et al., 2009). Deforestation due to population growth and industrialization is a serious problem as it threatens ecological balance.

The development of mathematical modeling contributes to tackling ecological

problems. Several experts have conducted research related to the depletion of forest resources. These include studies examining the effects of industry and pollution on forest resources (Dubey et al., 2002). Research on models of depletion of forest resources due to population, population pressure, and industrialization (Dubey et al., 2009). In 2011, there was an article analyzing models of renewable resource depletion due to population and industrialization, as well as the impact of technology on renewable resource conservation (Shukla et al., 2011).

Additionally, some studies have presented a forest resource depletion model that takes into account the impact of industrial area accumulation on the biomass of forest resources (Ramdhani et al., 2015). In 2018, a study expanded on the forest resource depletion model previously discussed by Ramdhani et al. by adding the factor of population growth rate resulting from migration due to the presence of industries (Anggriani et al., 2018). In the previous year, research has shown that there is a correlation between forest degradation caused by population density and industrial density (Sundar et al., 2017). In 2019 a model of forest destruction with industrial factors and fire which explain cause of forest destruction (Mohamad et al., 2019).

A study which analyze about mangrove forest resource depletion models with time delay (Wakhidah et al., 2022). Based on research conducted by Ramdhani et al. (2015), modifications to the model were made by considering the factor of forest biomass depletion due to the influence of population density and considering population growth factors due to forest biomass resources. The model also refers to the research of Dubey et al., (2009) and Shukla et al., (2011). The problem will be formulated in the form of a nonlinear mathematical model, and then the system stability will be analyzed. The stability analysis is done by analyzing the stability of the fixed points of the model. Then numerical simulations of the model are carried out to observe changes in forest resources due to population density. Additionally, the impact of forest resource biomass on population growth can also be observed.

METHOD

This study used a literature review method by studying several reference books, scientific literature, as well as the findings of previous research. This was done in order to establish a solid theoretical foundation and gather information pertinent to the research problem.

RESULT AND DISCUSSION

Before discussing the essence of this research, the following an explanation regarding the model that depletion forest biomass model.

1. **Forest Resource Depletion Model**

The forest resource depletion model makes several assumptions, namely:

a. The density of forest resource biomass and population density are influenced by the logistic equation.
b. The average growth of population pressure is proportional to population density.
c. Forest resource depletion is assumed to be a result of population and industrialization.
d. All parameters in the model have positive values.

The pattern of the forest resource depletion model schematically can be seen in the compartment diagram that has been presented (see Figure 1).

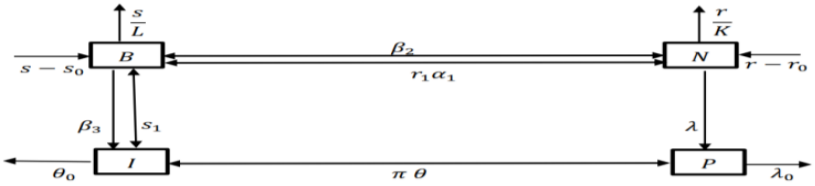


Figure 1. Compartment Diagram of Forest Resource Biomass Depletion Model

Based on the assumptions and compartment diagram, the forest resource depletion model is given in the form of a non-linear ordinary differential equation system:

$$\begin{aligned}\frac{dB}{dt} &= s\left(1-\frac{B}{L}\right)B - s_0B - \beta_2NB - s_1IB \\ &\quad -\beta_3B^2I - \alpha_1B^2N \\ \frac{dN}{dt} &= r\left(1-\frac{N}{K}\right)N - r_0N + \beta_1NB + \\ &\quad \alpha_1r_1B^2N, \\ \frac{dP}{dt} &= \lambda N - \lambda_0P - \theta I, \\ \frac{dI}{dt} &= \pi\theta P + \pi_1s_1IB - \theta_0I,\end{aligned} \quad (1)$$

$N(0) = N_0 \geq 0, B(0) = B_0 \geq 0, P(0) = P_0 \geq 0, I(0) = I_0 \geq 0, 0 \leq \pi \leq 1, 0 \leq \pi_1 \leq 1.$

In system (1), $B(t)$ represents the density of forest resource biomass, $N(t)$ represents the density of population, $P(t)$ represents the density of population pressure, and $I(t)$ represents the density of industrialization.The parameters are as follows (see Table 1).

Table 1. Description Parameters

Parameters	Description
s	the average intrinsic growth coefficient of forest biomass resources
L	the carrying capacity of forest biomass resources
s_0	the average coefficient of natural depletion of forest biomass resources
β_3	the coefficient of depletion of forest resources due to the accumulation of industrial areas
α_1	the coefficient of depletion of forest biomass resources due to the influence of population density
r_1	the coefficient of population growth rate due to forest biomass resources
r_0	the coefficient of the average natural death rate of the population
r	the coefficient of the average intrinsic growth of population density
K	the carrying capacity of population density
β_1	the average growth of the entire population density due to forest resources
β_2	the coefficient of the average relationship between the depletion of forest biomass resources and population density
λ	the coefficient of the average growth of population pressure
λ_0	the coefficient of the average depletion of population pressure naturally
θ	the coefficient of the average depletion of population pressure due to increasing industrialization
s_1	the coefficient of the average depletion of forest biomass density due to industrialization
$\pi_1 s_1$	the average growth of industrialization due to forest resources
π	the average growth of industrialization due to population pressure
θ_0	the coefficient of the average external control of industrialization

The solution of system (1) is a non-negative and bounded solution. This is demonstrated by the following Lemma 1:

Lemma 1. The following set $\Omega = \{(B,N,P,I): 0 \leq B \leq L, 0 \leq N \leq F_m, 0 \leq P \leq H_m, 0 \leq B+P+I \leq (\lambda F_m + sL)/\phi\}$ is a solution of system (1) where $F_m = \frac{K}{r}(r + \beta_1 L + \alpha_1 r_1 L^2)$, $H_m = \frac{\lambda F_m}{\lambda_0}$, and $\phi = \min((s_0, \lambda_0 - \theta\pi, \theta_0)$.

Proof. Based on system (1), we obtain

$$\frac{dB}{dt} \leq s\left(1-\frac{B}{L}\right)B \quad (2)$$

Since the rate of forest biomass density is non-negative, equation (2) can be written as

$$0 \leq \frac{dB}{dt} \leq s\left(1-\frac{B}{L}\right)B. \quad (3)$$

In (3) is solved by algebraic calculations and comparison theorem, it is obtained that

$$0 \leq B(t) \leq L. \quad (4)$$

Based on the second equation of system (1), we obtain:

$$\frac{dN}{dt} \leq r\left(1-\frac{N}{K}\right)N + \beta_1 NL + \alpha_1 r_1 L^2 N$$
$$= (r + \beta_1 L + \alpha_1 r_1 L^2)\frac{K}{r}$$
$$= F_m$$

which gives the form

$$0 \leq N(t) \leq \frac{K}{r}(r + \beta_1 L + \alpha_1 r_1 L^2) = F_m.$$

based on the third equation in system (1), it is obtained

$$\frac{dP}{dt} \leq \lambda F_m - \lambda_0 P$$

which takes the following form

$$0 \leq P(t) \leq \frac{\lambda F_m}{\lambda_0} = H_m.$$

Next, from the last equation of system (1), we perform a summation involving variables $B, P,$ and I

$$\frac{dB}{dt} + \frac{dP}{dt} + \frac{dI}{dt} \leq \lambda N + sB - s_0 B$$
$$- (1-\pi_1)s_1 IB + (\pi\theta - \lambda_0)P$$
$$- \theta_0 I$$
$$\leq \lambda F_m + sL - \phi(B + P + I)$$

The inequality obtained is:

$0 \leq B(t) + P(t) + I(t) \leq \frac{\lambda F_m + sL}{\phi}$

with$\phi = \min(s_0, \lambda_0 - \theta\pi, \theta_0)$. ■

2. Fixed Point Analysis

To find the fixed points of system (1), one needs to determine when $\frac{dB}{dt}$, $\frac{dN}{dt}$, $\frac{dP}{dt}$are equal to zero. By analyzing the model, four fixed points are found that are all non-negative: $E_1(0,0,0,0)$, $E_2(\hat{B},0,0,0)$, $E_3(0,\bar{N},\bar{P},\bar{I})$, and $E^*(B^*,N^*,P^*,I^*)$. The values of $\hat{B}$, $\bar{N}$, $\bar{P}$ and $\bar{I}$ are determined by the following equations: $\hat{B} = \frac{L(s-s_0)}{s}$, $\bar{N} = \frac{K(r-r_0)}{r}$, $\bar{P} = \frac{\lambda K\theta_0(r-r_0)}{r(\pi\theta^2+\theta_0\lambda_0)}$, and $\bar{I} = \frac{K\pi\theta\lambda(r-r_0)}{r(\pi\theta^2+\theta_0\lambda_0)}$. It should be noted that :$r > r_0$ and $s > s_0$.

The first fixed point, $E_1(0,0,0,0)$, represents the condition when there is no influence whatsoever from population density, population pressure, and industrialization on forest biomass density.

The existence of the fixed point E* is presented below. The first step is to set the right-hand side of system (1) to zero. The values of B*, N*, P*, and I* are obtained by solving the resulting equations, which give the following expressions:

$B^* = \frac{L}{s+L\beta_3 I^*+LN^*}(s - s_0 - \beta_2 N^* - s_1 I^*)$ (5a)

$N^* = \frac{K}{r-K\alpha_1 r_1 (B^*)^2}(r - r_0 + \beta_1 B^*)$ (5b)
$= f(B^*)$

$I^* = \frac{\pi\theta\lambda f(B^*)}{\lambda_0\theta_0+\pi\theta^2-\pi_1 s_1\lambda_0 B^*} = h(B^*)$ (5c)

$P^* = \frac{\lambda f(B^*)-\theta h(B^*)}{\lambda_0} = g(B^*).$ (5d)

The value of $I^* = h(B^*)$is positive only if $\lambda_0\theta_0 + \pi\theta^2 - \pi_1 s_1\lambda_0 B^* > 0$ then based on the first equation in system (1) and applying $B^*, N^*, P^*$and I^* to the equation, it can be rewritten as:

$$\frac{dB^*}{dt} = \big(s - s_0 - \beta_2 f(B^*) - s_1 h(B^*) -$$
$$\beta_3 B^* h(B^*) - \alpha_1 r_1 B^* f(B^*)\big)B^* - \frac{sB^{*2}}{L}.$$

where,

$$s - s_0 - \beta_2 f(B^*) - s_1 h(B^*) - \beta_3 B^* h(B^*)$$
$$-\alpha_1 r_1 B^* f(B^*)$$

Is the intrinsic biomass growth rate coefficient of forest resources. This will be positive for $B^* \geq 0$. Thus, for $B^* = 0$, we obtain:

$$s - s_0 - \beta_2\frac{K(r-r_0)}{r} - s_1 h(0) > 0 (6)$$

By substituting $N^* = f(B^*)$ and $I^* = h(B^*)$ into equation (5a), we obtain

$$B^* = \frac{L}{s+L\beta_3 h(B^*)+\mathrm{L}f(B^*)}(s - s_0 - \beta_2 f(B^*) -$$
$$s_1 h(B^*))$$
(7)

By using equation (7), $F(B^*)$ can be defined as follows:

$$F(B^*) = (s + L\beta_3 h(B^*)$$
$$+ \mathrm{L}f(B^*))\,B^* - L(s$$
$$- s_0 - \beta_2 f(B^*) - s_1 h(B^*)$$
$$= 0. (8)$$

Equation (8) can be expressed in the following form by utilizing equation (6) as follow:

$$F(0) = -L(s - s_0 - \beta_2 \frac{K(r - r_0)}{r} s_1 h(0)) < 0$$

$$F(L) = (\beta_3 h(L) + N)L^2 + L(sL + s_0 + \beta_2 f(L) - s_1 h(L) > 0.$$

If $F'(B)$ is greater than zero at point E^*, then this is a sufficient condition for B^* to be unique/special. This form can be obtained from equation (8)

$$F'(B) = s + L\beta_3 h'(B)B + L\beta_3 h(B) + \mathrm{L}f'(B) + L(\beta_2 f'(B) - s_1 h'(B) > 0$$

because f' and h' are positive values for $\lambda_0\theta_0 + \pi\theta^2 > \pi_1 s_1 \lambda_0 B^*$.

$$J_{E_i=} \begin{bmatrix} s\left(1 - \frac{2B}{L}\right) - s_0 - \beta_2 N - s_1 I - 2\beta_3 BI - 2\alpha_1 BN & -\beta_2 B - \alpha_1 B^2 & 0 & -s_1 B - \beta_3 B^2 \\ \beta_1 N + 2\alpha_1 r_1 BN & r\left(1 - \frac{2N}{K}\right) - r_0 + \beta_1 B + \alpha_1 r_1 B^2 & 0 & 0 \\ 0 & \lambda & -\lambda_0 & -\theta \\ \pi_1 s_1 I & 0 & \pi\theta & \pi_1 s_1 B - \theta_0 \end{bmatrix}$$

This indicates that there is a specific value of B^*, within the range of $0 \le B \le L$, for which $F(B^*) = 0$. Once the value of B^* is determined, equations (5b), (5c), and (5d) can be used to obtain the values of N^*, I^*,andP^*. Therefore, the existence of the fixed point $E^*(B^*, N^*, P^*, I^*)$is apparent.

Based on equation (5b), it can be seen that $N^* \to \frac{K}{r}(r - r_0)$ as $B^* \to 0$. This implies that the population density tends towards the environmental carrying capacity as the biomass density of the forest resource tends towards zero.

3. Analysis of Fixed-Point Stability

In this section, stability analysis of system (1) is carried out for each fixed point previously described. The stability of fixed points $E_1, E_2,$ and E_3 is analyzed using linearization method. After linearization of system (1), the Jacobian matrix of the system of equations will be obtained. Then, the Jacobian matrix is sought for each fixed point. The type of stability of fixed points can be seen based on the eigenvalue of their Jacobian matrix(Robinson, 2004). The stability analysis of fixed point E^* is discussed specifically using the Lyapunov stability theory.

Performing a linearization on system (1) will result in a Jacobi matrix (see Figure 2).

Figure 2.The Jacobian Matrix of The System(1)

Stability of Fixed Point $E_1(0,0,0,0)$

The Jacobian matrix in Figure 2 is used to linearize system (1) around the fixed point $E_1(0,0,0,0)$. The resulting Jacobian matrix can be expressed as:

$$J_{E_1} = \begin{bmatrix} s - s_0 & 0 & 0 & 0 \\ 0 & r - r_0 & 0 & 0 \\ 0 & \lambda & -\lambda\theta & -\theta \\ 0 & 0 & \pi\theta & -\theta_0 \end{bmatrix}$$

The eigenvalues of matrix J_{E_1} are determined by solving the equation$|J_{E_1} - \psi I| = 0$, where I is the identity matrix, results as:

$$\psi_1 = s - s_0 > 0$$
$$\psi_2 = r - r_0 > 0$$
$$\psi_3 = \frac{1}{2}(\theta_0 + \lambda_0) + \frac{1}{2}\sqrt{\theta_0^2 - 4\theta^2\pi + \lambda_0^2 - 2\lambda_0\theta_0}$$
$$\psi_4 = \frac{1}{2}(\theta_0 + \lambda_0) - \frac{1}{2}\sqrt{\theta_0^2 - 4\theta^2\pi + \lambda_0^2 - 2\lambda_0\theta_0}$$

From these results, it can be seen that the eigenvalues ψ_1 and ψ_2 are positive. Therefore, the fixed point $E_1(0,0,0,0)$ is an unstable, if $(\theta_0 + \lambda_0) \pm \sqrt{\theta_0^2 + \lambda_0^2 - 2\lambda_0\theta_0 - 4\theta^2\pi} < 0$,

then this point is saddle point and unstable (Robinson, 2004).

Stability of Fixed Point $E_2(\hat{B}, 0, 0, 0)$

The linearization at the fixed point $E_2(\hat{B}, 0, 0, 0)$ yields a Jacobian matrix (see Figure 3).

$$J_{E_2} = \begin{bmatrix} s - s_0 - \frac{2s\hat{B}}{L} & -\hat{B}(\beta_2 + \alpha_1\hat{B}) & 0 & \hat{B}(-s_1 + \beta_3\hat{B} \\ 0 & r - r_0 + \hat{B}(\beta_1 + r_1\alpha_1\widehat{B}) & 0 & 0 \\ 0 & \lambda & -\lambda_0 & -\theta \\ 0 & 0 & \pi\theta & \pi_1 s_1\hat{B} - \theta_0 \end{bmatrix}$$

Figure 3. Jacobian Matrix of Fixed Point $E_2(\hat{B}, 0, 0, 0)$

Furthermore, the eigenvalues are obtained by solving the characteristic equation $|J_{E_2} - \psi I| = 0$, where I is the identity matrix as follows:

$$\psi_1 = s_0 - s < 0,$$

$$\psi_2 = r - r_0 + \hat{B}(\beta_1 + \hat{B}r_1\alpha_1) > 0$$

$$\psi_{3,4} = \frac{1}{2}(\pi_1 s_1\hat{\mathrm{B}} - \theta_0 - \lambda_0) \pm \frac{1}{2}\Big((\pi_1 s_1\hat{\mathrm{B}})^2 - 2(\pi_1 s_1\hat{\mathrm{B}}\theta_0 - \pi_1 s_1\hat{\mathrm{B}}\lambda_0 + \theta_0\lambda_0 + 2\theta^2\pi) + {\theta_0}^2 + {\lambda_0}^2\Big)^{\frac{1}{2}}.$$

By examining the eigenvalues, it is shown that the eigenvalue $\psi_1 < 0$and the eigenvalue eigen $\psi_2 > 0$. Therefore, the fixed point $E_2(\hat{B}, 0, 0, 0)$ is a saddle point and unstable (Robinson, 2004).

Stability of Fixed Point$E_3(\mathbf{0}, \bar{N}, \bar{P}, \bar{I})$

Linearization at fixed point $E_3(0, \bar{N}, \bar{P}, \bar{I})$ results in a Jacobi matrix (see Figure 4).

$$J_{E_3} = \begin{bmatrix} s - \beta_2\bar{N} - s_1\bar{I} - s_0 & 0 & 0 & 0 \\ \beta_1\bar{N} & r - r_0 & 0 & 0 \\ 0 & \lambda & -\lambda\theta & -\theta \\ \pi_1 s_1 I & 0 & \pi\theta & -\theta_0 \end{bmatrix}$$

Figure 4. Jacobian Matrix of Fixed PointE_3

The eigenvalues of the matrix J_{E_3} are determined by solving the equation $|J_{E_3} - \psi I| = 0$, where I is the identity matrix:

$$\psi_1 = s - s_0 - \beta_2\bar{N} - s_1\bar{I}$$

$$\psi_2 = r_0 - r + \frac{2\bar{N}}{K}$$

$$\psi_3 = \frac{1}{2}(-\lambda\theta - \theta_0) + \frac{1}{2}\sqrt{\lambda_0^2 - 2\lambda_0\theta_0 + \theta_0^2 - 4\theta^2\pi}$$

$$\psi_4 = \frac{1}{2}(-\lambda\theta - \theta_0) - \frac{1}{2}\sqrt{\lambda_0^2 - 2\lambda_0\theta_0 + \theta_0^2 - 4\theta^2\pi}$$

Based onthe eigenvalues, if the following condition is satisfied:

$$(-\lambda_0 - \theta_0) \pm \sqrt{\lambda_0^2 - 2\lambda_0\theta_0 + \theta_0^2 - 4\theta^2\pi} < 0,$$
$$s - s_0 - \beta_2\bar{N} - s_1\bar{I} > 0,$$
$$r_0 - r + \frac{2\bar{N}}{K} < 0$$

then the fixed point $E_3(0, \bar{N}, \bar{P}, \bar{I})$ is a saddle point fixed point and is unstable (Robinson, 2004).

Stability of Fixed Point $E^*(B^*, N^*, P^*, I^*)$

In the previous section, the stability analysis of the first equation fixed points has been described. It is difficult to describe the stability of $E^*(B^*, N^*, P^*, I^*)$ like the three previous fixed points. Hence, based on the concept of Lyapunov stability, the fowing theorem provides the conditions that must be satisfied for E^* to be locally asymptotically stable.

Theorem 1. E^* is a locally asymptotically stable fixed point if the following conditions are satisfied:

$$s_1\pi_1 B^* < \theta_0,$$
$$\frac{\pi\beta_1(\beta_3 B^* + s_1)}{\pi_1\beta_2 s_1 I^*} < \frac{8\lambda_0 r}{\lambda^2 K}.$$

Proof. To prove Theorem 1, the first step is to linearize the system of equations (1) by substituting the following equation:

$$B = B^* + b,\ N = N^* + n, P = P^* + p, I = I^* + i, \quad (9)$$

The variables b, n, p, i represent small disturbances around the fixed point $E^*(B^*, N^*, P^*, I^*)$. Furthermore, a positive definite function is introduced as follows, and it is defined that $V: D \subset R^4 \to R$:

$$V(b,n,p,i) = \frac{1}{2}\left(\frac{C_1 b^2}{B^*} + \frac{C_2 n^2}{N^*} + C_3 p^2 + C_4 i^2\right) \quad (10)$$

By selecting positive constants C_i for $i = 1, 2, 3, 4$ that satisfy equation (10)(Dubey et al., 2009), the function in equation (10) is considered a Lyapunov function due to satisfying properties described (Khalil, 2002):

1. it is a positive definite function.
2. The following discussion will demonstrate that $\dot{V}$ is a negative definite function

The time derivative of V for system (1) is formulated as follows:

$$\frac{dV}{dt} = V_b\frac{dB}{dt} + V_n\frac{dN}{dt} + V_p\frac{dP}{dt} + V_i\frac{dI}{dt}. \quad (11)$$

Next, equations (9) and the differential equation in system (1) are substituted into equation (11) to obtain:

$$\frac{dV}{dt} = \frac{C_1 b}{B^*}\left[s\left(1 - \frac{B^*+b}{L}\right)(B^* + b) - s_0(B^* + b) - \beta_2(N^* + n)(B^* + b) - s_1(I^* + i)(B^* + b) - \beta_3(B^* + b)^2(I^* + i) - \alpha_1(B^* + b)^2(N^* + n)\right] + \frac{C_2 n}{N^*}\left[r\left(1 - \frac{N^*+n}{K}\right)(N^* + n) - r_0(N^* + n) + \beta_1(N^* + n)(B^* + b) + \alpha_1 r_1(B^* + b)^2(N^* + n)\right] + C_3 p\,[\,\lambda(N^* + n) - \lambda_0(P^* + p) - \theta(I^* + i)] + C_4 i\,[\,\pi\theta(P^* + p) + \pi_1 s_1(I^* + i)(B^* + b) - \theta_0(I^* + i)] \quad (12)$$

It is possible to express equation (12) in the following form:

$$\frac{dV}{dt} = \frac{C_1 b}{B^*}\left[s(B^* + b) - \frac{s}{L}(B^* + b)^2 - s_0(B^* + b) - \beta_2(N^* + n)(B^* + b) - s_1(I^* + i)(B^* + b) - \beta_3(B^* + b)^2(I^* + i) - \alpha_1(B^* + b)^2(N^* + n)\right] + \frac{C_2 n}{N^*}\left[r(N^* + n) - \frac{r}{K}(N^* + n)^2 - r_0(N^* + n) + \beta_1(N^* + n)(B^* + b) + \alpha_1 r_1(B^* + b)^2(N^* + n)\right] + C_3 p\,[\,\lambda(N^* + n) - \lambda_0(P^* + p) - \theta(I^* + i)] + C_4 i\,[\,\pi\theta(P^* + p) + \pi_1 s_1(I^* + i)(B^* + b) - \theta_0(I^* + i)] \quad (13)$$

Equation (13) is expanded and obtained asfollows:

$$\frac{dV}{dt} = C_1 sb + \frac{C_1 s}{B^*} b^2 - \frac{C_1 s B^*}{L} b - \frac{2C_1 s}{L} b^2 - C_1 s_0 b - \frac{C_1 s_0}{B^*} b^2 - C_1 \beta_2 N^* b - \frac{C_1 \beta_2 N^*}{B^*} b^2 - C_1 \beta_2 bn - \frac{C_1 \beta_2}{B^*} b^2 n - C_1 s_1 I^* b - \frac{C_1 s_1 I^*}{B^*} b^2 - C_1 s_1 bi - C_1 \beta_3 B^* I^* b - C_1 \beta_3 B^* bi - 2C_1 \beta_3 I^* b^2 - 2C_1 \beta_3 b^2 i - C_1 \beta_3 I^* b^3 - \frac{C_1 \beta_3}{B^*} b^3 i + C_2 rn + \frac{C_2 r}{N^*} n^2 - \frac{C_2 r N^*}{K} n - \frac{2C_2 r}{K} n^2 - \frac{C_2 r}{KN^*} n^3 - C_2 r_0 n - \frac{C_2 r_0}{N^*} n^2 + C_2 \beta_1 B^* n + C_2 \beta_1 bn + \frac{C_2 \beta_1 B^*}{N^*} n^2 + \frac{C_2 \beta_1}{N^*} bn^2 + C_3 \lambda N^* p + C_3 \lambda np - C_3 \lambda_0 P^* p - C_3 \lambda_0 p^2 - C_3 \theta I^* p - C_3 \theta pi + C_4 \pi \theta P^* i + C_4 \pi \theta pi + C_4 \pi_1 s_1 I^* B^* i + C_4 \pi_1 s_1 I^* bi + C_4 \pi_1 s_1 B^* i^2 + C_4 \pi_1 s_1 bi^2 - C_4 \theta_0 I^* i - C_4 \theta_0 i^2. \quad (14)$$

From equation (14), the derivative of the Lyapunov function V with respect to time t for the linearized system (1) is obtained as follows:

$$\frac{dV}{dt} = -\frac{2C_1 s}{L} b^2 - C_1 \beta_2 bn - C_1 s_1 bi - C_1 \beta_3 B^* bi - 2C_1 \beta_3 I^* b^2 - \frac{2C_2 r}{K} n^2 - C_4 \theta_0 i^2 + C_2 \beta_1 bn + C_3 \lambda np - C_3 \lambda_0 p^2 - C_3 \theta pi + C_4 \pi \theta pi + C_4 \pi_1 s_1 I^* bi + C_4 \pi_1 s_1 B^* i^2 \quad (15)$$

The equation (15) can be rewritten in the following equation:

$$\frac{dV}{dt} = -C_1 \left(\frac{2s}{L} + 2C_1 \beta_3 I^*\right) b^2 - \frac{2C_2 r}{K} n^2 - C_3 \lambda_0 p^2 - C_4 (\theta_0 - \pi_1 s_1 B^*) i^2 + (C_2 \beta_1 - C_1 \beta_2) bn + (C_4 \pi_1 s_1 I^* - C_1 s_1 - C_1 \beta_3 B^*) bi + C_3 \lambda np + (C_4 \pi \theta - C_3 \theta) pi \quad (16)$$

The values of C_i where $i = 1, 2, 3, 4$ will be determined to reduce equation (16). The first step is to set the coefficient of bn equal to zero, which can be written as:

$$C_2 \beta_1 - C_1 \beta_2 = 0,$$

Choose $C_2 = 1$, so it is obtained:

$$C_1 \beta_2 = \beta_1$$

$$C_1 = \frac{\beta_1}{\beta_2}$$

Subsequently, the same procedure is carried out for the coefficients of bi, which can be expressed as:

$$C_4 \pi_1 s_1 I^* - C_1 s_1 - C_1 \beta_3 B^* = 0, \quad (17)$$

Then, substitute the value of C_1 into equation (17) to obtain:

$$C_4 \pi_1 s_1 I^* = \frac{\beta_1 (s_1 + \beta_3 B^*)}{\beta_2},$$

$$C_4 = \frac{\beta_1 (s_1 + \beta_3 B^*)}{\beta_2 \pi_1 s_1 I^*}. \quad (18)$$

Next, π is also set to zero, resulting in the following equation:

$$C_4 \pi \theta - C_3 \theta = 0, \quad (19)$$

Substituting the value of C_4 in equation (18) into equation (19), the value of C_3 is obtained as follows:

$$C_3 \theta = C_4 \pi \theta,$$

$$C_3 = \frac{\beta_1 \pi (s_1 + \beta_3 B^*)}{\beta_2 \pi_1 s_1 I^*}. \quad (20)$$

Therefore, equation (16) can be reduced to the following form

$$\frac{dV}{dt} = -C_1 \left(\frac{2s}{L} + 2C_1 \beta_3 I^*\right) b^2 - \frac{2r}{K} n^2 - C_3 \lambda_0 p^2 - C_4 (\theta_0 - \pi_1 s_1 B^*) i^2 + C_3 \lambda np. \quad (21)$$

where, C_i for $i = 1,2,3$ is a positive constants

The equation (21) will be negative definite if the following conditions are satisfied (Boyce& Diprima, 2012):

$$\theta_0 - \pi_1 s_1 B^* > 0,$$

$$s_1 \pi_1 B^* < \theta_0.$$

So, all coefficients b^2, n^2, p^2, and i^2 in equation (21) are already negative. The next step is to solve the last term of the equation using the concept of negative definiteness, $b^2 - 4ac < 0$ where(Boyce & DiPrima, 2012):

$$a = -\frac{2r}{K},$$
$$b = C_3\lambda,$$
$$c = -C_3\lambda_0,$$

The values of $a, b,$ and c above are substituted into the inequality below:

$$b^2 - 4ac < 0,$$

$$(C_3\lambda)^2 - 4\left(-\frac{2r}{K}\right)(-C_3\lambda_0) < 0,$$

so,

$$(C_3\lambda)^2 < 4\left(\frac{-2r}{K}\right)(-C_3\lambda_0),$$

Since λ^2 and C_3 are positive, the following inequality is obtained

$$C_3 < \frac{8\lambda_0 r}{\lambda^2 K}, \quad (22)$$

by substituting the value of C_3 from equation (20) into inequality (22), we obtain:

$$\frac{\pi\beta_1(\beta_3 B^* + s_1)}{\pi_1\beta_2 s_1 I^*} < \frac{8\lambda_0 r}{\lambda^2 K}.$$

Thus, under the following conditions:

$$s_1\pi_1 B^* < \theta_0$$

$$\frac{\pi\beta_1(\beta_3 B^* + s_1)}{\pi_1\beta_2 s_1 I^*} < \frac{8\lambda_0 r}{\lambda^2 K}$$

The fixed point $E^*(B^*, N^*, P^*, I^*)$ is a locally asymptotically stable. Therefore, Theorem 1 is proven.

Numeric Simulation

One of the aims of this research is to simulate the model to observe the effects of parameter value changes. Simulation is conducted since the system is difficult to observe directly, thus through simulation, the dynamics of the model can be observed. The initial parameter values for the simulation are (Dubey et al., 2009):

$L = 40,\ K = 50,\ \beta_3 = 6,\ \pi = 0.001,\ \theta = 8,$ $\lambda = 5,\ \beta_1 = 0.01,\ \beta_2 = 7, \pi_1 = 0.005,\ s_1 = 4, , s = 34,\ s_0 = \theta_0 = 1,\ r = 11,\ r_0 = 10.$ r_1=1 , α_1=0,00001, $\lambda_0 = 4$

Based on those parameter values, the value of the fixed point $E^*(B^*, N^*, P^*, I^*)$ is obtained:
$B^* = 0.69265,\ N^* = 4.57696,\ P^* = 5.62986,$ and $I^* = 0.45672$ (see Figure 5).

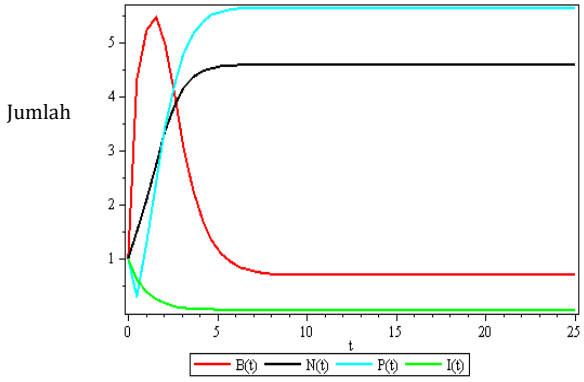


Figure 5. Existence of Fixed PointE^*

From Figure 5, it is observed that the solution of the system of equations (1) tends towards the fixed point $E^*(B^*, N^*, P^*, I^*)$. The forest biomass density converges to its equilibrium value of 0.69265, while the population density also converges to its equilibrium value of 4.57696. The pressure of population density approaches its equilibrium value of 5.62986, and the industrialization density approaches its equilibrium value of 0.45672. Therefore, based on the graph in Figure 5 and the condition of Theorem 1 being satisfied, it is concluded that the model in system (1) is locally asymptotically stable at the fixed point $E^*(B^*, N^*, P^*, I^*)$.

The following images show the simulation results with various parameter values that were previously given (see Figure 6).

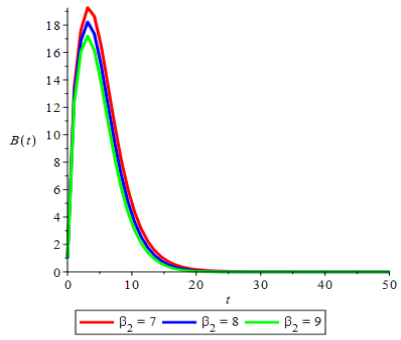


Figure 6. Changes in $B(t)$ Over Time$t$ with Varying Values of β_2

The graph in Figure 6 shows that the density of forest biomass (B) decreases until it reaches equilibrium as the value of β_2 increases. This means that the density of forest biomass decreases due to an increase in the rate of forest biomass depletion caused by the population growth. The equilibrium value of forest biomass density is lower than the carrying capacity of the environment for forest biomass.

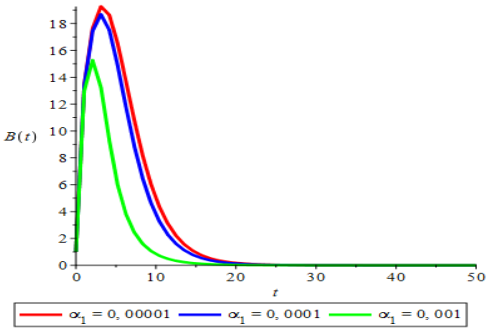


Figure 7**.** Change in $B(t)$ over Time$t$ with Varying Values of α_1.

Figure 7 illustrates that an increase in the value of α_1 leads to a decrease in $B(t)$ until it reaches its stability level. This implies that if there is an increase in the depletion of forest biomass due to the population density effect, the density of forest biomass resource will decrease until it reaches its stability level.

In Figure 8, it is illustrated that an augmentation in the value of β_1 leads to an elevation in $N(t)$ until it reaches a stable point. This implies that the density of the population grows until it attains a stable level as a result of the existence of forest biomass..

Figure 9 illustrates that as the value of λ increases, there is an increase in $P(t)$ towards its stable value. This means that if there is an increase in the growth rate of population pressure, then the population pressure will also increase until it reaches its equilibrium level.

In Figure 10, it is clear that $I(t)$ increases as π increases until it reaches its stable condition. This indicates that the density of industrialization will increase if the growth rate of industrialization due to population pressure increases until it reaches its equilibrium level.

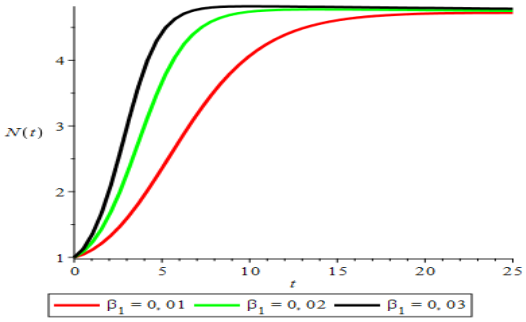


Figure 8. Changes in $N(t)$ Over Time $t$with Varying Values ofβ_1.

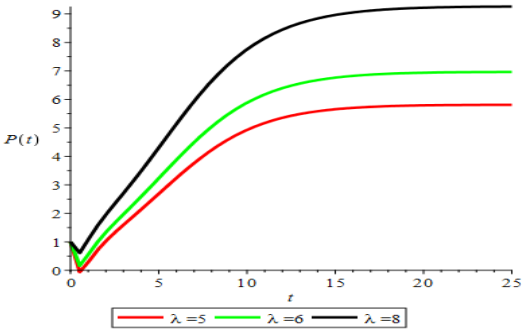


Figure 9. Changes in $P(t)$Over Time $t$with Varying Values of λ.

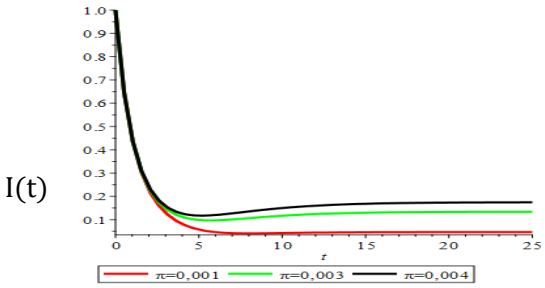


Figure 10. Change in $I(t)$ Over Time $t$with Varying Values of π.

CONCLUSION

The article discusses the analysis of a nonlinear mathematical model to study the depletion of forest resource biomass due to population growth, population pressure, and industrialization. The model considers the depletion of forest biomass due to population density and the growth of the population due to forest biomass resources. The model has been analyzed using stability theory of ordinary nonlinear differential equation systems and numerical simulations were performed. The following conclusions were obtained:

1. The model analysis shows that the non-linear mathematical model of forest resource biomass depletion has four non-negative fixed points, where the first three fixed points, namely $E_1(0,0,0,0)$, $E_2(\hat{B},0,0,0)$, $E_3(0,\bar{N},\bar{P},\ \bar{I})$, are saddle point and unstable if certain conditions are met. For the fourth fixed point, $E^*(B^*,N^*,P^*,I^*)$, it is a stable asymptotic local if it satisfies Theorem 1.
2. Based on the simulation results, it was found that the density of forest resources biomass decreases until it reaches its equilibrium point due to the increase in population, population pressure, and industrialization. The equilibrium level of forest biomass is smaller than the environmental carrying capacity of the forest resources.

3. Based on the simulation results, it was observed that the growth of the population leads to a decline in the biomass of forest resources. Conversely, forest biomass has a positive effect on the population level.

Based on all observed simulations, it was found that if stability conditions are met, the system dynamics always reach their equilibrium level.

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